J. Fluid Mech. (2006), vol. 565, pp. 95–103. © 2006 Cambridge University Press
doi:10.1017/S0022112006001984 Printed in the United Kingdom

Multiple-scale perturbation analysis of slowly evolving turbulence

By STEPHEN L. WOODRUFF[1] AND ROBERT RUBINSTEIN[2]
[1]Center for Advanced Power Systems, Florida State University, Tallahassee, FL 32310, USA
[2]Computational Aerosciences Branch, NASA Langley Research Center, Hampton, VA 23681, USA

(Received 26 March 2006 and in revised form 8 July 2006)

A multiple-scale perturbation expansion is applied to extract a closed system of two equations governing the scalar descriptors of the turbulence energy spectrum from a spectral closure model. The result applies when the length scale and total energy input of a force that maintains a steady state of homogeneous isotropic turbulence are perturbed slowly and the energy spectrum consequently evolves slowly compared to the time scales of the turbulence itself.

1. Introduction

Two-equation modelling is motivated by Kolmogorov's theory of the universal small scales in turbulent flows, which implies that a statistically steady state of isotropic turbulence is determined by its kinetic energy k and energy flux ϵ. The beautiful and useful consequence is that the infinite number of scales of motion in a turbulent flow can be described by only two parameters.

But Kolmogorov's theory strictly applies only to a steady state. To obtain practically useful results, two additional assumptions are generally introduced in deriving models. The first is that *a time-dependent turbulent state can be described by the time-varying parameters* $k(t)$ *and* $\epsilon(t)$, provided that these parameters vary slowly relative to the time scales of the turbulent fluctuations. This assumption seems justifiable in a very broad range of applications. But a second assumption departs much farther from the Kolmogorov theory and is far less obviously justifiable: it asserts that *the parameters* $k(t)$ *and* $\epsilon(t)$ *satisfy closed equations of motion.* No fundamental considerations justify this assertion, which is made simply to permit the formulation of models.

This paper treats this second assumption as a theoretical question: given that the energy spectrum can be described by two parameters, can it be demonstrated that these parameters satisfy closed equations of motion, and if not in general, under what conditions? Our purpose is to decide whether these equations exist; their formulation if they do, and their utility for modelling are separate problems that we do not address. The question will be posed in the simplest possible case of statistically non-stationary homogeneous isotropic turbulence. Despite its apparent simplicity, this provides a good test problem, because two-equation modelling is most likely to be valid in this case, if it is valid at all.

A two-equation description of slow spectral evolution will be derived for a particular spectral closure: the Heisenberg model (Batchelor 1953). The relevant properties of this model are that it is consistent with a Kolmogorov steady state and that it provides a dynamic theory of spectral evolution due to perturbations about this state.

It permits the study, in an analytically simple setting, of the question of when the coupled evolution of an infinite number of scales of motion can be replaced by two equations.

The derivation is based on an expansion for the energy spectrum about a base state using multiple-scale perturbation methods; in the present study, the base state is homogeneous isotropic turbulence maintained by steady forcing, as in the two-scale formalism of Yoshizawa (1998). The parameters that define the base state are allowed to vary slowly; solvability conditions that allow the perturbation expansion to proceed to higher order are used to derive evolution equations for the slowly varying parameters. The perturbation parameter will be identified with the ratio of the large-eddy turnover time to the rate of change of the turbulence statistics; in agreement with intuitive expectations, this ratio must be small in order for reduced-order modelling to be possible.

The analysis naturally recovers the energy balance as one equation. The second equation that emerges is neither a dissipation-rate transport equation of the standard form (Jones & Launder 1972), nor a moment evolution equation of the type sought by Laporta (1995), Besnard *et al.* (1996), Clark (1999) and others. In contrast to these references and other attempts to derive a two-equation model theoretically assuming a time-dependent Kolmogorov spectrum (for example, Yakhot & Smith 1992; Rubinstein & Zhou 1996), the present analysis focuses on the role of perturbations to the base state. Such perturbations are described by the energy transfer linearized about the base state; the null space of the adjoint of the linearized energy transfer is shown to determine the number and nature of the equations in the model. The second equation obtained by this procedure depends on the choice of the base state: selecting a self-similar time-dependent state, for example decaying isotropic turbulence, homogeneous shear flow or one of the infinite number of self-similar states identified for homogeneous isotropic turbulence in Rubinstein & Clark (2005), will lead to a different model. Thus, while our conclusions support finite-dimensional modelling of slowly varying turbulence, they contradict the idea that such modelling can be universal.

2. Derivation of a two-equation model from the Heisenberg closure

The general spectral evolution equation for homogeneous isotropic turbulence is (Batchelor 1953)

$$\frac{\partial}{\partial t}E(\kappa,t) = P(\kappa,t) - T(\kappa,t) - D(\kappa,t), \tag{2.1}$$

where $E(\kappa,t)$ is the energy spectrum, $P(\kappa,t)$ is the production spectrum, $T(\kappa,t)$ is the nonlinear energy transfer, and $D(\kappa,t)=2\nu\kappa^2E(\kappa,t)$ is the dissipation spectrum. Conservation of energy by nonlinear interaction implies that $\int_0^\infty \mathrm{d}\kappa\ T(\kappa,t)=0$. It is convenient to introduce the *spectral energy flux* $\mathscr{F}(\kappa,t)=\int_0^\kappa \mathrm{d}\mu\ T(\mu,t)$ where $\mathscr{F}(0,t)=\mathscr{F}(\infty,t)\equiv 0$, in terms of which (2.1) takes the conservation form

$$\frac{\partial}{\partial t}E(\kappa,t) = P(\kappa,t) - \frac{\partial \mathscr{F}}{\partial \kappa}(\kappa,t) - D(\kappa,t). \tag{2.2}$$

In the Heisenberg closure, the flux $\mathscr{F}$ is the functional of the energy spectrum E,

$$\mathscr{F}[E(\kappa,t)] = C_H\alpha(\kappa,t)\beta(\kappa,t) = C_H\int_0^\kappa \mathrm{d}\mu\ \mu^2E(\mu,t)\int_\kappa^\infty \mathrm{d}p\ E(p,t)\theta(p,t) \tag{2.3}$$

with the time-scale closure $\theta(p,t)=[\sqrt{p^3E(p,t)}]^{-1}$. The first factor α is the square of the strain due to modes with wavenumbers less than κ, and β is a viscosity formed from modes with wavenumbers greater than κ. The constant C_H can be chosen to match the (empirically given) Kolmogorov constant, but its value will be irrelevant in this analysis.

The Heisenberg model is the simplest closure that captures the role of nonlinear interactions over a continuous range of scales of motion in turbulent energy transfer; it can be obtained (Kraichnan 1987) as a limit of a much more sophisticated closure, the Kraichnan (1971) test-field model. Although more refined models of energy transfer exist, the Heisenberg model permits an analysis of the possibility of replacing an infinite coupled system by a finite number of equations in an analytically tractable setting, and can help us understand when this replacement is possible, how many equations are necessary, and how we can find these equations if they exist.

The starting point is steady-state homogeneous isotropic turbulence driven by statistically steady isotropic forcing. This state is defined by the time-independent form of (2.1):

$$P(\kappa)-T(\kappa)-D(\kappa)=0. \tag{2.4}$$

The production spectrum is assumed to be concentrated near some dominant scale L_P and to be characterized completely by L_P and the total rate of energy production, $\bar{P}=\int_0^\infty \mathrm{d}\kappa \;\; P(\kappa)$. The production spectrum therefore admits the two-parameter description $P(\kappa)=P(\kappa;\bar{P},L_P)$, where different pairs $(\bar{P},L_P)$ are understood to give different spectra $P(\kappa)$. The functional form of $P(\kappa;\bar{P},L_P)$ will be considered fixed throughout.

In agreement with the Kolmogorov theory, the Heisenberg model predicts an energy spectrum $E(\kappa)$ that can be characterized for wavenumbers $\kappa \ll \kappa_d$, the Kolmogorov scale, by its dissipation rate ϵ and by a spectral integral scale L; the definition of L is not unique, but it will be convenient to assume that is defined so that in a steady state, $L=L_P$. In a steady state, $\bar{P}=\int_0^\infty \mathrm{d}\kappa \;\; D(\kappa)=\epsilon$, therefore we have

$$P(\kappa)=P(\kappa;\bar{P},L_P)=P(\kappa;\epsilon,L), \quad E(\kappa)=E(\kappa;\epsilon,L) \quad \text{with} \quad \epsilon=\bar{P}, \quad L=L_P, \tag{2.5}$$

so that the steady energy spectrum has the same two-parameter structure as the production spectrum, with coincident scalar descriptors. The two-parameter description of the energy spectrum assumes that we are in the high-Reynolds-number limit, in which $\kappa_d\to\infty$, $\nu\to 0$, while $\epsilon\sim\nu^3\kappa_d^4$ remains constant. In this limit, $D(\kappa)$ can be ignored in (2.4) and the energy balance maintained by setting $\mathscr{F}(\infty)=\epsilon$.

2.1. *Perturbation expansion and solvability condition*

We wish to pass from this static state to a temporally evolving state. Regardless of any special properties of the time evolution, one equation must always be satisfied: the equation for the kinetic energy $k(t)=\int_0^\infty \mathrm{d}\kappa \;\; E(\kappa,t)$ found by integrating (2.2) over all κ:

$$\frac{\partial k}{\partial t}=\bar{P}(t)-\epsilon(t). \tag{2.6}$$

We will now define a special type of temporal evolution appropriate for multiple-scale perturbation theory. We assume that $P(\kappa,t)$ and $E(\kappa,t)$ can still be characterized by the parameters $\bar{P},L_P,\epsilon,L$, but that these parameters all vary slowly in time; thus, $L_P=L_P(\tau)$, $\bar{P}=\bar{P}(\tau)$, $L=L(\tau)$, $\epsilon=\epsilon(\tau)$, where $\tau=\delta t$ with δ a small parameter that will be characterized shortly. Consider a slowly varying time-dependent reference state described by $\epsilon_0(\tau)$ and $L_0(\tau)$, and assume that the last two equalities in (2.5) that link the production spectrum and the energy spectrum are only weakly perturbed, so that $\bar{P}(\tau)-\epsilon_0(\tau)$ and $L_P(\tau)-L_0(\tau)$ are both of order δ: we write these conditions in the normalized form $\Delta\bar{P}(\tau)\equiv[\bar{P}(\tau)-\epsilon_0(\tau)]/\delta=O(1)$ and $\Delta L_P(\tau)\equiv[L_P(\tau)-L_0(\tau)]/\delta=O(1)$ so that

$$\bar{P}=\epsilon_0+\delta\Delta\bar{P}, \quad L_P=L_0+\delta\Delta L_P. \tag{2.7}$$

In order to permit these perturbations to develop, the steady production spectrum in (2.5) is replaced by

$$P(\kappa,t)=P_0(\kappa;\epsilon_0(\tau),L_0(\tau))+\delta P_1(\kappa;\bar{P}(\tau),L_P(\tau);\epsilon_0(\tau),L_0(\tau)) \tag{2.8}$$

where we have written $P_0(\kappa;\epsilon_0,L_0)$ for $P(\kappa;\epsilon_0,L_0)$ to emphasize that we are perturbing about the reference state with parameters (ϵ_0,L_0). Then the energy spectrum admits the corresponding expansion

$$E(\kappa,t)=E_0(\kappa;\epsilon_0(\tau),L_0(\tau))+\delta E_1(\kappa;\bar{P}(\tau),L_P(\tau);\epsilon_0(\tau),L_0(\tau))+\cdots. \tag{2.9}$$

Although no particular form of the production perturbation P_1 will be specified, we will assume that this perturbation vanishes if $\bar{P}=\epsilon$ and $L_P=L$; thus, $P_1(\kappa;\epsilon,L;\epsilon,L)\equiv 0$ and Taylor series expansion shows that δP_1 depends on $\bar{P}$ and L_P through the $O(\delta)$ differences $\bar{P}-\epsilon_0$ and L_P-L_0. For example, we could set δP_1 to the linear part of the Taylor series expansion of $P(\kappa;\bar{P},L_P)-P(\kappa;\epsilon_0,L_0)$ about $P(\kappa;\epsilon_0,L_0)$ in powers of $\bar{P}-\epsilon_0$ and L_P-L_0.

Following the standard method of multiple-scale perturbation analysis, (2.8) and (2.9) impose slow temporal variation of $P(\kappa,t)$ and $E(\kappa,t)$ by requiring that they depend on time through a slow time variable. The expansion (2.9) induces expansions of single-point moments $k=k_0+\delta k_1+\cdots$ and $\epsilon=\epsilon_0+\delta\epsilon_1+\cdots$ where $k_i=\int_0^\infty \mathrm{d}\kappa\, E_i(\kappa)$ and $\epsilon_i=\int_0^\infty \mathrm{d}\kappa\, 2\nu\kappa^2 E_i(\kappa)$. Similarly, since regardless of how L is defined, it is a functional of E, we also have the expansion $L=L_0+\delta L_1+\cdots$. If the slow evolution of turbulence quantities is such that $(\partial L/\partial\tau)/L\sim(\partial\epsilon/\partial\tau)/\epsilon\sim\epsilon/k$, then $(\partial L/\partial t)/L\sim(\partial\epsilon/\partial t)/\epsilon\sim\delta(\epsilon/k)$, and therefore $\delta\sim(\dot{\epsilon}/\epsilon)(k/\epsilon)\sim(\dot{L}/L)(k/\epsilon)$, where the overdot denotes $\partial/\partial t$. Therefore, the expansion parameter δ is the ratio of the large-eddy turnover time and the characteristic time over which the statistics vary.

Substituting (2.9) in (2.2) leads to the quasi-steady relation

$$P_0-\partial\mathscr{F}[E_0]/\partial\kappa-D_0=0 \tag{2.10}$$

for E_0 at lowest order. This equation is satisfied because the time-dependent parameters in E_0 coincide with those in P_0; thus, to lowest order, the energy spectrum instantaneously adjusts to the time-dependent production spectrum. At the next order, we obtain an equation for E_1:

$$\frac{\partial}{\partial\kappa}\mathscr{L}[E_1(\kappa,\tau)]=-\frac{\partial}{\partial\tau}E_0(\kappa;\epsilon_0(\tau),L_0(\tau))+P_1(\kappa,\tau)-D_1(\kappa,\tau). \tag{2.11}$$

The complete argument lists in E_1 and P_1 are understood, but have been suppressed for conciseness. Note that the time derivative of E_0, ignored at leading order in (2.10), appears as part of the inhomogeneous term in an equation for E_1; this is consistent with treating the time dependence of E_0 as a small perturbation. The linear operator $\mathscr{L}$ in (2.11) is the Fréchet derivative of $\mathscr{F}$ at E_0:

$$\mathscr{L}[E_1(\kappa,\tau)] = -\tfrac{1}{2}C_H\alpha(\kappa,\tau)\int_\kappa^\infty \mathrm{d}p\ \beta'(p,\tau)\frac{E_1(p,\tau)}{E_0(p,\tau)} + C_H\beta(\kappa,\tau)\int_0^\kappa \mathrm{d}\mu\ \alpha'(\mu,\tau)\frac{E_1(\mu,\tau)}{E_0(\mu,\tau)} \qquad (2.12)$$

where the primes denote derivatives with respect to the wavenumber argument. The appearance of D_1 on the right-hand side of (2.11) will be justified subsequently.

Equation (2.11) can only have a solution if its right-hand side is orthogonal to the null space of the linear operator on the left-hand side. This solvability condition (or conditions; there will be as many conditions as there are dimensions of the null space) provides equations for the slowly varying quantities ϵ and L. The adjoint of the linear operator on the left-hand side of (2.11) is $\mathscr{L}^\dagger\partial/\partial\kappa$ with the adjoint of $\mathscr{L}$ given by

$$\mathscr{L}^\dagger[\Phi(\kappa,\tau)] = C_H\frac{\alpha'(\kappa,\tau)}{E_0(\kappa,\tau)}\int_\kappa^\infty \mathrm{d}p\ \beta(p,\tau)\Phi(p,\tau) - \tfrac{1}{2}C_H\frac{\beta'(\kappa,\tau)}{E_0(\kappa,\tau)}\int_0^\kappa \mathrm{d}\mu\ \alpha(\mu,\tau)\Phi(\mu,\tau). \qquad (2.13)$$

Note that E_1 satisfies homogeneous boundary conditions $E_1(0) = E_1(\infty) = 0$.

2.2. *Solution of the adjoint equation*

We are seeking solutions Ψ of the homogeneous equation

$$\mathscr{L}^\dagger\left[\frac{\partial\Psi}{\partial\kappa}\right] = 0. \qquad (2.14)$$

One solution is immediately obvious: $\Psi \equiv 1$. We will show that the equation $\mathscr{L}^\dagger[\Phi] = 0$ has exactly one more solution; the general solution of (2.14) is therefore $\Psi = A\Psi_1 + B\Psi_2$, where $\Psi_1 = 1$, and $\Psi_2 = \int_0^\kappa \mathrm{d}p\ \Phi(p)$.

The solution Φ is constructed by writing (2.14) using (2.13) as

$$\frac{\alpha'(\kappa)}{\beta'(\kappa)}M(\kappa) - \tfrac{1}{2}N(\kappa) = 0, \qquad (2.15)$$

with $M(\kappa) = \int_\kappa^\infty \mathrm{d}\mu\ \beta(\mu)\Phi(\mu)$ and $N(\kappa) = \int_0^\kappa \mathrm{d}p\ \alpha(p)\Phi(p)$. Using $N' = -(\alpha/\beta)\ M'$, the derivative of (2.15) may be written $M' + W(\kappa)M = 0$, where

$$W(\kappa) = \frac{(\alpha'(\kappa)/\beta'(\kappa))'}{\alpha'(\kappa)/\beta'(\kappa) + \tfrac{1}{2}\alpha(\kappa)/\beta(\kappa)}. \qquad (2.16)$$

This equation has the solution $M(\kappa) = \exp(-\int_0^\kappa \mathrm{d}p\ W(p))$. Then $\Phi(\kappa) = -M'(\kappa)/\beta(\kappa)$ and

$$\Psi_2(\kappa) = \int_0^\kappa \mathrm{d}\mu\ \frac{W(\mu)}{\beta(\mu)}\exp\left(-\int_0^\mu \mathrm{d}p\ W(p)\right). \qquad (2.17)$$

2.3. *Application of solvability conditions*

Solvability conditions are now obtained by multiplying (2.11) by the functions Ψ_1 and Ψ_2 and integrating over all κ. Since Ψ_1 and Ψ_2 are solutions of the adjoint equation, the left-hand side will be zero after this integration; the result can therefore be written as

$$\int_0^\infty \mathrm{d}\kappa\ \Psi_i(\kappa)\frac{\partial}{\partial\tau}E_0(\kappa,\epsilon(\tau),L(\tau)) = \int_0^\infty \mathrm{d}\kappa\ \Psi_i(\kappa)[P_1(\kappa,\tau) - D_1(\kappa,\tau)], \quad i = 1, 2. \qquad (2.18)$$

Here and henceforth, the subscript 0 will be suppressed in ϵ and L, as we will only solve for the lowest-order quantities ϵ_0 and L_0. Without loss of generality, E_0 may be represented by appealing to dimensional analysis as

$$E_0(\kappa;\epsilon(\tau),L(\tau)) = \epsilon(\tau)^{2/3}L(\tau)^{5/3}\hat{\phi}(\kappa L(\tau)), \tag{2.19}$$

in terms of a dimensionless function $\hat{\phi}$; this expression coincides with the Besnard *et al.* (1996) spectral ansatz. Note that

$$\frac{\partial}{\partial\tau}E_0(\kappa,\epsilon(\tau),L(\tau)) = \tfrac{2}{3}\epsilon(\tau)^{-1/3}L(\tau)^{5/3}\hat{\phi}(\kappa L(\tau))\frac{\partial\epsilon}{\partial\tau} + (\epsilon(\tau)L(\tau))^{2/3}\left(\tfrac{5}{3}\hat{\phi}(\kappa L(\tau)) + (\kappa L(\tau))\hat{\phi}'(\kappa L(\tau))\right)\frac{\partial L(\tau)}{\partial\tau}. \tag{2.20}$$

The production perturbation on the right-hand side of (2.18) is also represented in a normalized form as

$$P_1(\kappa;\bar{P}(\tau),L_P(\tau);\epsilon(\tau),L(\tau)) = \epsilon(\tau)\hat{p}_1(\kappa L(\tau);\bar{P}(\tau),L_P(\tau);\epsilon(\tau),L(\tau)). \tag{2.21}$$

Substituting (2.20) and (2.21) in (2.18), we obtain

$$\epsilon^{-1/3}L^{2/3}I_i^1(\epsilon,L)\frac{\partial\epsilon}{\partial\tau} + (\epsilon L)^{2/3}I_i^2(\epsilon,L)\frac{\partial L}{\partial\tau} = \epsilon I_i^3(\bar{P},L_P;\epsilon,L) - \int_0^\infty \mathrm{d}\kappa\ \Psi_i(\kappa L)D_1(\kappa,\tau), \quad i=1,2, \tag{2.22}$$

where

$$\left.\begin{aligned} I_i^1(\epsilon,L) &= \frac{2}{3}\int_0^\infty \mathrm{d}p\ \Psi_i(p)\hat{\phi}(pL),\\ I_i^2(\epsilon,L) &= \int_0^\infty \mathrm{d}p\ \Psi_i(p)(5/3\hat{\phi}(pL) + (\kappa L)\hat{\phi}'(pL)),\\ I_i^3(\bar{P},L_P;\epsilon,L) &= \int_0^\infty \mathrm{d}p\ \Psi_i(p)\hat{p}_1(pL;\bar{P},L_P;\epsilon,L). \end{aligned}\right\} \tag{2.23}$$

We claim that both terms containing D_1 in (2.22) vanish in the high-Reynolds-number limit. Assuming tentatively that this is true, (2.22) is replaced by

$$\epsilon^{-1/3}L^{2/3}I_i^1(\epsilon,L)\frac{\partial\epsilon}{\partial\tau} + (\epsilon L)^{2/3}I_i^2(\epsilon,L)\frac{\partial L}{\partial\tau} = \epsilon I_i^3(\bar{P},L_P;\epsilon,L), \quad i=1,2. \tag{2.24}$$

Equation (2.24) with the definitions (2.23) is the required two-equation model. It describes the slow variation of the spectral parameters $\epsilon(\tau)$ and $L(\tau)$ due to slow changes of production through $\bar{P}(\tau)$ and slow changes of the forcing length scale through $L_P(\tau)$. The term inside parentheses in the definition of I_i^2 vanishes on a Kolmogorov spectrum and these integrals are therefore dominated by contributions from the largest scales. As noted earlier, the production perturbation depends on the differences $\bar{P}(\tau)-\epsilon(\tau)$ and $L_P(\tau)-L(\tau)$; thus, the driving terms in this model appear in I_i^3 through the differences $\Delta\bar{P}$ and ΔL_P. Since the production perturbation P_1 has been assumed to vanish if $\Delta\bar{P}=\Delta L_P=0$, or $\bar{P}=\epsilon$ and $L_P=L$, I_1^3 and I_2^3 both vanish under these conditions.

We remark that the solution of (2.11) satisfies $E_1 \propto \dot{\epsilon}/\epsilon$. Dimensional analysis therefore gives $E_1 \sim (\dot{\epsilon}/\epsilon)\epsilon^{1/3}\kappa^{-7/3}$, so that E_1 exhibits the $\kappa^{-7/3}$ scaling proposed by Yoshizawa (1994) for time-dependent turbulence. This scaling can also be expressed as $E_1 \sim (\dot{\epsilon}/\epsilon)[\epsilon^{1/3}\kappa^{2/3}]^{-1}E_0$. Thus, $E_1/E_0 \sim (\dot{\epsilon}/\epsilon)/(\epsilon^{1/3}\kappa^{2/3})$, the ratio of the turnover time at wavenumber κ to the time scale of evolution of ϵ. Then $E_1/E_0 \sim (\dot{\epsilon}/\epsilon)(k/\epsilon)(\kappa L)^{-2/3}$;

the maximum value of this ratio occurs when $\kappa L \approx 1$, confirming that the expansion parameter of the theory is indeed $(\dot{\epsilon}/\epsilon)(k/\epsilon)$ (Rubinstein & Clark 2005).

It remains to justify dropping the D_1 terms in (2.22). For the first equation, in which $\Psi_1 \equiv 1$, the viscous term represents a correction to the leading-order dissipation, $\epsilon_1 = 2\nu \int_0^\infty d\kappa \kappa^2 E_1$. We can estimate this term using the result just noted that $E_1 \sim \kappa^{-7/3}$. Since its integrand is of order $\kappa^{-1/3}$, the integral defining ϵ_1 will be dominated by large wavenumbers. Then $\epsilon_1 \sim \nu \kappa_d^{2/3} \sim \kappa_d^{-2/3} \sim \nu^{1/2}$, where κ_d is the Kolmogorov scale; therefore the correction to the dissipation vanishes in the limit $\nu \to 0$. It will be shown in the next section that $\Psi_2 \sim \kappa^{-4}$ at large κ. In this case, the integral $2\nu \int_0^\infty d\kappa \Psi_2 \kappa^2 E_1$ converges at large wavenumbers; since it is therefore of order ν, this integral also vanishes in the high-Reynolds-number limit.

2.4. *General structure of the model*

Our goal was to find conditions, if any exist, under which a two-equation model can be justified, using standard multiple-scale perturbation theory. Nevertheless, it may be useful to note some general properties of the result. Since $\Psi_1 = 1$, the first solvability condition is the kinetic energy balance; (2.22) with $i = 1$ states this balance in terms of the variables L and ϵ; (2.6) would have emerged directly if the spectrum had been expressed in terms of k and ϵ instead of in terms of L and ϵ.

The consequence of this analysis is that $\epsilon(\tau)$ and $L(\tau)$, which together characterize the leading-order energy and production spectra, are found by solving the compatibility conditions; neither is known in advance. This structure is familiar in multiple-scale perturbation analysis, but at variance with standard models, which treat the production as a known driving force. In the present model, it is the increments $\Delta \bar{P}$ and ΔL_P that are arbitrary, subject only to the requirements of slow temporal variation and small amplitude imposed by the multiple-scale analysis. The total production and production length scale are recovered through (2.7).

The explicit formulation of the second compatibility equation requires knowing the function Ψ_2, but since the expression for Ψ_2 in (2.17) is rather complicated, it may be helpful to consider its behaviour in an inertial range. Assuming Kolmogorov scaling, $\alpha \sim (3/4)\kappa^{4/3}$ and $\beta \sim (3/4)\kappa^{-4/3}$; consequently, $W \sim (16/3)\kappa^{-1}$ and $\Psi_2 \sim \kappa^{-4}$. This strong concentration of Ψ_2 at large scales justifies ignoring the viscous term in (2.22) when $i = 2$. This equation is therefore very sensitive to the integral scale, and is essentially independent of the inertial range.

3. Conclusions

The present derivation of a two-equation model that describes slow spectral evolution near a steady state of isotropic turbulence places the understanding of such modelling in the properties of the linearized energy transfer $\mathscr{L}$, and identifies its adjoint, $\mathscr{L}^\dagger$ as the key object: the dimension of its null-space determines the number of equations in the model and its null vectors detemine the analytical structure of the relevant equations. From this viewpoint, it is evident that the analysis can be applied to more realistic closure theories like the direct interaction approximation (Kraichnan 1959), test-field model (Kraichnan 1971), or Lagrangian renormalized approximation (Kaneda 1981).

Existing empirical models (Reynolds 1976) assume a general form for the two-equation model, which is calibrated to reproduce the behaviour of some simple self-similar flows. Without in any way deprecating the practical utility of such models, we note that this procedure leaves completely unanswered the question of whether the result can be validly applied away from the self-similar calibration cases. In assuming that it can, the empirical approach makes the unsubstantiated assumption that turbulence is itself a kind of 'fluid' with the same 'properties' in all flows. In contrast, the present approach provides genuine dynamic equations for ϵ and L without postulating self-similarity or requiring calibration; the equations are restricted only by the assumptions of small-amplitude, slowly varying perturbations that underly the perturbation expansion.

Previous theoretical treatments of the dissipation-rate transport equation have been based either on a Kolmogorov spectrum (for example, Laporta 1995; Yakhot & Smith 1992; Rubinstein & Zhou 1996) or on an exactly self-similar state (for example, Besnard *et al.* 1996; Clark 1999). From the viewpoint of our perturbation theory expressed in (2.9), assuming a Kolmogorov state means that the energy spectrum is characterized by E_0 alone. Then integration of (2.10) simply gives $\bar{P} = \epsilon_0$. No other conclusion is possible if the perturbation expansion stops at E_0; we therefore believe these theoretical analyses must be inconclusive. The present analysis instead emphasizes the role of the perturbations to the local Kolmogorov spectrum E_0 in statistically non-stationary turbulence that were first identified by Yoshizawa (1994). This point is also discussed by Rubinstein & Clark (2005).

Theoretical treatments based on self-similarity have tended to emphasize that the validity of models is in fact restricted to self-similar flows (Clark 1999), where the model is understood to state certain special scaling invariance properties (Clark & Zemach 1998) rather than general equations of motion. A fundamental difficulty then arises, because an infinite number of self-similar states is possible (Rubinstein & Clark 2005). In principle, the present analysis could be applied to perturbations about self-similar time-dependent states, but it should be noted that the multiple-scale perturbation theory based on a time-dependent reference state is considerably more complicated.

The model is derived by a multiple-scale perturbation expansion with some formal analogies to the Chapman–Enskog expansion of kinetic theory. Thus, the leading-order spectrum $E_0(\kappa, \epsilon(\tau), L(\tau))$ is a 'local Kolmogorovian' analogous to the local Maxwellian of kinetic theory, although here we focus on temporal rather than spatial variation. In both cases, the failure of the 'local' solution to satisfy the governing equations generates the perturbation expansion. The Fréchet derivative $\mathscr{L}$ is analogous to the linearized collision integral, but because steady-state turbulence is far from thermal equilibrium, this operator is not self-adjoint. However, the main result, a compatibility equation, is of course common to general multiple-scale perturbation expansions.

We conclude by noting two possible extensions of this work. The discussion of closure theories naturally suggests the question of whether a similar analysis might apply to simulation data without the intervention of closure. Closure expresses the energy transfer in terms of the energy spectrum; we cannot assume this relation in simulation data. A simplification is provided by the (statistical) assumption that all moments depend on time only through two slowly varying properties. This assumption may lead to a closure for the increments of statistics, which is all that is required by this type of analysis. Obviously however, this remains a topic for future investigation.

The restriction of this analysis to temporal non-stationarity with spatial homogeneity leaves important modelling issues unaddressed. Slow and fast time variables were introduced in the multiple-scale analysis, but it is likely that the analysis could be extended to inhomogeneous flows by introducing large- and small-scale spatial variables as in the two-scale direct interaction approximation formalism of Yoshizawa

(1984, 1998). From the viewpoint of perturbation theory, this simple approach is open to the objection that it assumes weak inhomogeneity at the outset. A more general approach might begin with a general inhomogeneous closure theory like that of Kraichnan (1972), and seek spatial analogues of the temporal local Kolmogorov spectrum used in this paper.

REFERENCES

Batchelor, G. K. 1953 *The Theory of Homogeneous Turbulence*. Cambridge University Press.

Besnard, D., Harlow, F., Rauenzahn, C. & Zemach, C. 1996 Spectral transport model for turbulence. *Theor. Comput. Fluid Dyn.* **8**, 1.

Clark, T. T. 1999 Two-point closures and statistical equilibrium. In *Modeling Complex Turbulent Flows* (ed. M. D. Salas, J. N. Hefner & L. Sakell), p. 183. Kluwer.

Clark, T. T. & Zemach, C. 1998 Symmetries and the approach to statistical equilibrium in isotropic turbulence. *Phys. Fluids* **10**, 2846.

Jones, W. P. & Launder, B. E. 1972 The prediction of laminarization with a two-equation model of turbulence. *Intl J. Heat Mass Transfer* **15**, 301.

Kaneda, Y. 1981 Renormalized expansions in the theory of turbulence with the use of the Lagrangian position function. *J. Fluid Mech.* **107**, 131.

Kraichnan, R. H. 1959 The structure of isotropic turbulence at very high Reynolds numbers. *J. Fluid Mech.* **5**, 497.

Kraichnan, R. H. 1971 Almost-Markovian Galilean-invariant turbulence model. *J. Fluid Mech.* **47**, 513.

Kraichnan, R. H. 1972 Test-field model for inhomogeneous turbulence. *J. Fluid Mech.* **56**, 287.

Kraichnan, R. H. 1987 An interpretation of the Yakhot-Orszag turbulence theory. *Phys. Fluids* **30**, 2400.

Laporta, A. 1995 Étude spectrale et modélisation de la turbulence inhomogène. Dissertation, Lyon I.

Reynolds, W. C. 1976 Computation of turbulent flows. *Annu. Rev. Fluid Mech.* **8**, 183.

Rubinstein, R. & Clark, T. T. 2005 Self-similar turbulence evolution and the dissipation rate transport equation. *Phys. Fluids* **17**, 095104.

Rubinstein, R. & Zhou, Y. 1996 Analytical theory of the destruction terms in dissipation rate transport equations. *Phys. Fluids* **8**, 3172.

Yakhot, V. & Smith, L. M. 1992 The renormalization group, the ϵ-expansion, and derivation of turbulence models. *J. Sci. Comput.* **7**, 35.

Yoshizawa, A. 1984 Statistical analysis of the deviation of the Reynolds stress from its eddy-viscosity representation. *Phys. Fluids* **27**, 1377.

Yoshizawa, A. 1994 Nonequilibrium effect of the turbulent energy production process on the inertial range energy spectrum. *Phys. Rev.* E **49**, 4065.

Yoshizawa, A. 1998 *Hydrodynamic and Magnetohydrodynamic Turbulent Flows: Modelling and Statistical theory*. Kluwer.